$C^k(\Omega) \overset{\text{def}}{=} \{ f : \Omega \to \mathbb{R}$ has up to kth order continuous derivatives $\}$ $k \geq 0$

e.g. $C^0(\Omega) = C(\Omega) = \{$ conti. func on $\Omega \}$

$C^1(\Omega) = \{$ conti. func on Ω with conti derivative $\}$

Def: A sym matrix $A \in \mathbb{R}^{n\times n}$ is called <u>positive semi-definite</u> if $\forall u \in \mathbb{R}^n$,

$$(u, Au) \overset{\text{def}}{=} u \cdot (Au) = u^T A u \geq 0$$

Prop. (1) If $f \in C^1(\Omega)$, then

f is convex $\Leftrightarrow$ $f(y) \geq f(x) + \nabla f^T(x)(y-x)$ $\forall x, y \in \Omega$

(2) If $f \in C^2(\Omega)$, then

f is convex $\Leftrightarrow$ the Hessian matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{n\times n}$

is positive semi-definite.

Lemma: If h is a convex func, then $\{x \in D(h) : h(x) \leq 0\}$ is a convex set.

Proof: Set $B \overset{\text{def}}{=} \{x \in D(h) : h(x) \leq 0\}$.

Need to prove: $\forall x, y \in B$, $\forall t \in [0,1]$

$tx + (1-t)y \in B$.

i.e. $h(tx + (1-t)y) \leq 0$

To show this, we know $h(x) \leq 0$, $h(y) \leq 0$.

Since h is convex,

$$h(tx+(1-t)y) \leq t\,h(x) + (1-t)\,h(y)$$

$$\leq t \cdot 0 + (1-t) \cdot 0$$

$$= 0$$

By the def of B, we have

$$tx+(1-t)y \in B. \quad \blacksquare$$

Def: Let $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$

A func $g(x) = a^T x + b$ is called an <u>affine func</u>.

Lemma: If g is an affine func, then the set $\{x \in D(g) : g(x) = 0\}$

is a convex set.

Proof: Set $B \overset{\text{def}}{=} \{x \in D(g) : g(x) = 0\}$

Need to prove: $\forall x, y \in B \quad \forall t \in [0,1]$

$$\Rightarrow \quad tx + (1-t)y \in B$$

i.e. $\quad g(tx + (1-t)y) = 0$

To show this, we know $g(x) = g(y) = 0$

Since g is affine, $g(x) = a^T x + b$.

so $g(tx + (1-t)y)$ $\qquad tb + (1-t)b$

$$= a^T(tx + (1-t)y) + b$$

$$= t(a^T x + b) + (1-t)(a^T y + b)$$

$$= t\,g(x) + (1-t)\,g(y)$$
$$= t \cdot 0 + (1-t) \cdot 0$$
$$= 0. \qquad \blacksquare$$

Def: An optimization problem of the form

$$\min f(x)$$

subject to $\quad h_i(x) \le 0 \qquad i = 1, \dots, I$

$$g_j(x) = 0 \qquad j = 1, \dots, J$$

is called a <u>convex optimization problem</u>

if $f, h_1, \dots, h_I$ are convex func
and $g_1, \dots, g_J$ are affine func.

Remark: The optimization problem is equivalent to

$$\min_{D} f(x)$$

where $D \overset{\text{def}}{=} D(f) \cap \{x : h_i(x) \leq 0 \quad i = 1, \dots I\}$

$\cap \{x : g_j(x) = 0, \ j = 1, \dots, J\}$

For a convex optimization problem,

then D is convex.