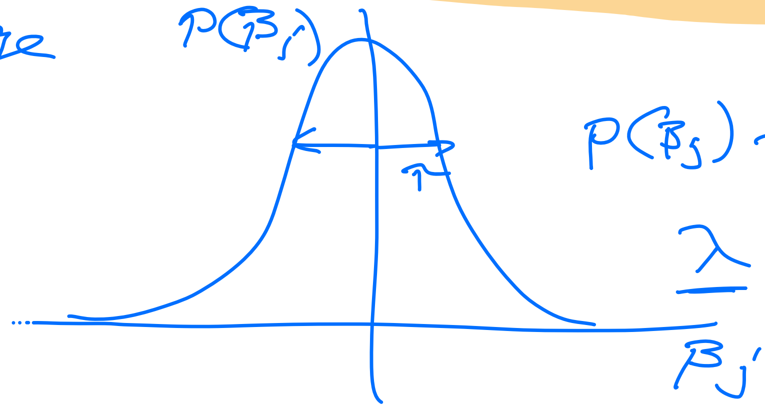
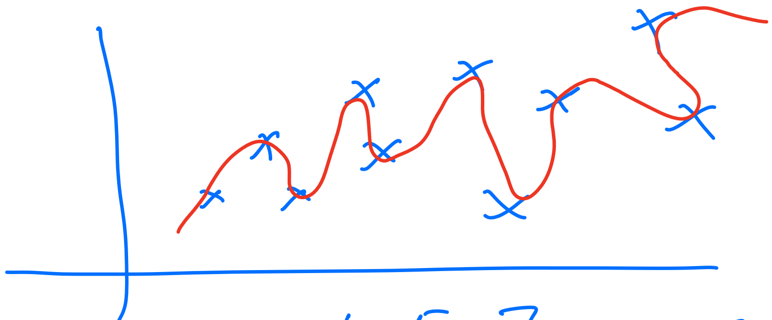
Lecture September 16

Ridge



$$P(\beta_j) \sim e^{-\beta_j^2/2\tau^2}$$

$$\underline{\lambda} = \frac{1}{2\tau^2}$$

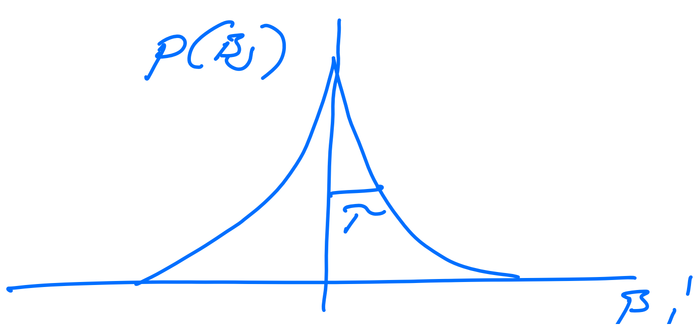


$\mathbb{E}[\underline{\beta_j}] \sim$ large

$\mathrm{var}[\beta_j] \sim$ large

Lasso

$$P(\beta_j) \sim e^{-|\beta_j|/\tau}$$



Central-limit theorem

$X \sim$ iid

$p(x)$

$$\bar{X} = \underline{\mu} = \int p(x)\,dx\,x \quad \left(\sum_{i=1}^{n} p(x_i)\,x_i\right)$$

$$\bar{X} \to \underline{\bar{X}_i} \to \underline{X_i}$$

$$Z = \frac{x_1 + x_2 + \ldots x_m}{m}$$

$$= \frac{1}{m}\sum_{i=1}^{m} \bar{x}_i = \frac{1}{m}\sum_{i=1}^{m} \frac{1}{n}\sum_{j=1}^{n} x_{i,j}$$

what is $\bar{p}(z)$

$$p(x_1 x_2 \ldots x_m) = p(x_1)\,p(x_2) \ldots p(x_m)$$

$$\bar{p}(z) = \int p(x_1)\,dx_1 \int p(x_2)\,dx_2 - \ldots$$

$$\int p(x_m)\,dx_m$$

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$$\times\, \delta\left(z - \frac{x_1 + x_2 + \cdots + x_m}{m}\right)$$

$$\delta\left(z - \frac{x_1 + x_2 + \cdots + x_m}{m}\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dq\, \exp\left[iq\left(z - \frac{x_1 + x_2 + \cdots + x_m}{m}\right)\right]$$

$$\bar{p}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(iq(z-\mu))\,dq$$

$$\times \left[\int_{-\infty}^{\infty} dx\, p(x)\, \exp\left(iq\,\frac{(\mu - x)}{m}\right)\right]^m$$

$$\boxed{\begin{aligned} \mu &= \int p(x)\, x\, dx \qquad \int p(x)\,dx = 1 \\ \sigma^2 &= \int p(x)(x-\mu)^2\, dx \end{aligned}}$$

$$\int_{-\infty}^{\infty} dx\, p(x)\, \exp\left(iq\left(\frac{\mu - x}{m}\right)\right)$$

$-\int dx\, p(x)\, x = \mu$

$$= \int_{-\infty}^{\infty} dx\, p(x) \left[1 + iq\frac{(\mu - x)}{m} - q^2 \frac{(\mu - x)^2}{2m^2} + \dots \right]$$

$$= 1 + 0 - \frac{q^2 \sigma^2}{2m^2} + \dots$$

$m \to \infty$

$$\bar{p}(z) = \frac{1}{\sqrt{2\pi \sigma^2/m}} \exp\left[\frac{-(z-\mu)^2}{2(\sigma/m)^2} \right]$$

variance $= \sigma^2/m$

$\Rightarrow$ standard deviation

$\sigma/\sqrt{m}$

Confidence intervals

$$\mathrm{Prob}(a \le x \le b)$$

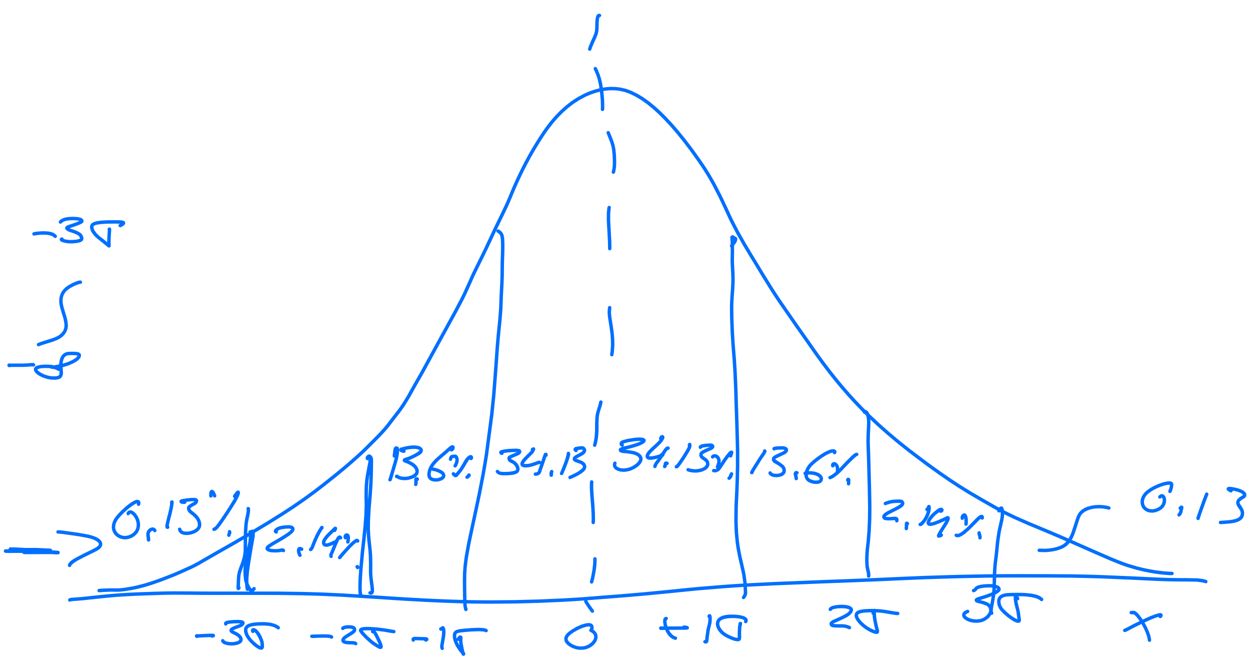
$$= \int_a^b p(x)\,dx$$

$$\int_{x \in D} p(x)\,dx = 1$$

cumulative probability:

$$P_i = \int_a^{x_i} p(x)\,dx$$

$$\int_{-\infty}^{-3\sigma}$$



$$P(-\sigma < 0 < \sigma) \cong 68\%$$

$$\boxed{P(-2\sigma < \sigma < 2\sigma) \simeq 95\%}$$

$$P(\beta - z\sigma, \beta + z\sigma)$$

$$2\sigma \sim z = 1.96 \simeq 2$$

$$\mathbb{E}[\beta_j] = \beta_j$$

$$var[\beta_j] = \sigma^2 (X^T X)^{-1}_{jj}$$

(OLS)

pinv

Resampling methods

- Bootstrap:
  (scikit-learn: resample(x, y))

Algorithm

sample $X = [x_0, x_1, x_2 .. x_{n-1}]$

(i) Draw a Bootstrap sample $[x_0^*, x_1^*, x_2^*, \ldots x_{n-1}^*]$ (By placing back)

compute $\hat{\beta}_n^* = g(x_0^*, x_1^*, \ldots x_{n-1}^*)$

(ii) Repeat the previous step B-times yielding the estimator $\hat{\beta}_{n,1}^*, \hat{\beta}_{n,2}^*, \ldots \hat{\beta}_{n,B}^*$

(iii) compute the variance

$$\hat{S} = \sqrt{\frac{1}{B} \sum_{j=1}^{B} \left(\hat{\beta}_{n,j}^* - \bar{\beta} \right)^2}$$

$$\bar{\beta} = \frac{1}{B} \sum_{j=1}^{B} \hat{\beta}_{n,j}^*$$

(iv) output $\hat{S}$