FYS-STK3155/4155, lecture October 20, 2025

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$$f(x) = \exp(x^2) = b = \exp(a)$$

$$a = x^2$$

$$f'(x) = 2 \cdot x \cdot b$$

$$\frac{df}{dx} = \frac{df}{db}\,\frac{db}{da}\,\frac{da}{dx}$$

$$\frac{df}{db} = 1, \quad \frac{db}{da} = \exp(a), \quad \frac{da}{dx} = 2x$$

$$x^2 \qquad \exp(a)$$

$$x \longrightarrow a \longrightarrow b \longrightarrow f$$

Example

$$f(x) = \sqrt{x^2 + \exp(x^2)} \quad 5 \text{ FLOPS}$$

$$f'(x) = \frac{x(1+\exp x^2)}{\sqrt{x^2+\exp(x^2)}}$$

10 FLOPS

$a = x^2 \wedge b = \exp x^2 = \exp a$

$c = a + b \wedge d = f(x) = \sqrt{c}$

$$\frac{da}{dx} = 2x \quad \wedge \quad \frac{db}{da} = \exp a$$

$$\frac{dc}{da} = \underline{1} \quad \wedge \quad \frac{dc}{db} = \underline{1}$$

$$\frac{dd}{dc} = \frac{1}{2\sqrt{c}} \quad \wedge \quad \frac{df}{dd} = \underline{1}$$

$$\frac{df}{dc} = \frac{df}{dd}\frac{dd}{dc} = \frac{1}{2\sqrt{c}}$$

$$\frac{df}{db} = \frac{df}{dc}\frac{dc}{db} = \frac{1}{2\sqrt{c}}$$

$$\frac{df}{da} = \frac{df}{dc}\underbrace{\frac{dc}{da}}_{=1} + \frac{df}{db}\underbrace{\frac{db}{da}}_{=1}$$

$$= \frac{1+\exp(a)}{2\sqrt{c}}$$

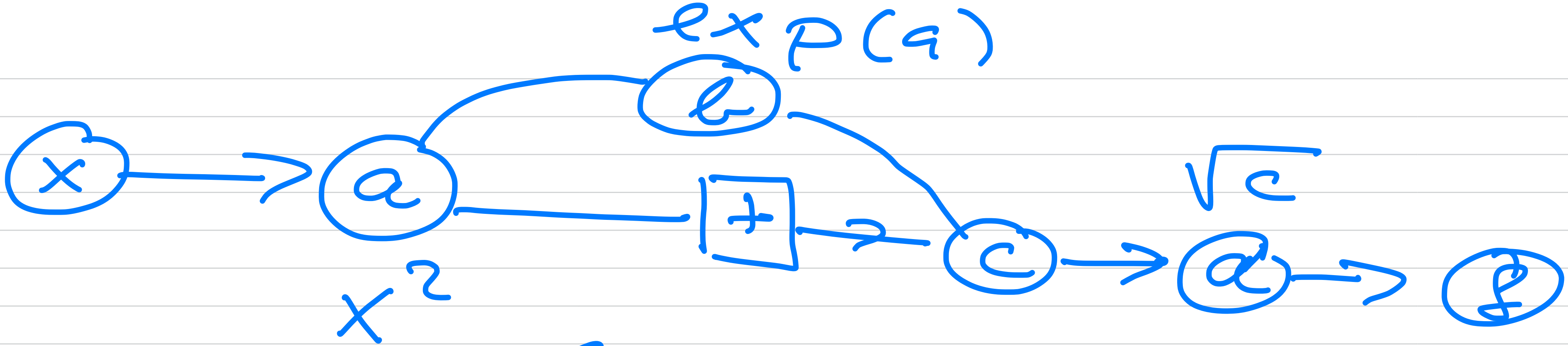
4 Flops

$$f = \sqrt{a+b}$$

$$= d = \sqrt{c}$$

$$\frac{df}{dx} = \frac{df}{da}\underbrace{\frac{da}{dx}}_{=x\cdot 2}$$

$$= \frac{x\left(1+\exp(a)\right)}{\sqrt{c}} = \boxed{\frac{x(1+e)}{d}}$$



$$c = a + b$$

$$d = \sqrt{a+b} = \sqrt{c}$$

algorithm

Define $\cancel{x} = x_1 \ldots x_\ell$ inputs

and $\underbrace{x_{\ell+1} \ldots x_L}_{\text{intermediate variables}}$ $\quad x_L = f$

in our example $l = \underline{1}$
only one input

$x_1 = x$; $x_2 = a = x^2$; $x_3 = b = \exp a$

; $x_4 = c = a + b$; $x_5 = d = f(x)$

$= \sqrt{c}$

$i = l+1, \dots, L$ (here $i = 2, 3, 4, 5$)

we define elementary functions of the variables

x_i $\quad x_3 = g_3(x_{pa}(x_3)) = \exp(a)$

We can compute the gradients by back propagating the derivatives using the chain rule

$$\frac{\partial f}{\partial x_2} = 1 = \frac{\partial f}{d d}$$

$$\frac{\partial f}{\partial x_i} = \sum_{x_j;\, x_i = Pa(x_j)} \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial x_i}$$

Example $x_4 = c$

$$\frac{\partial f}{\partial x_4} = \frac{\partial f}{\partial c} = \underbrace{\frac{\partial f}{\partial d}}_{= \frac{\partial f}{\partial x_1}} \underbrace{\frac{\partial d}{\partial c}}_{= \frac{1}{2\sqrt{c}}}$$

$$\frac{\partial f}{\partial x_3} = \frac{\partial f}{\partial b} = \overbrace{\frac{\partial f}{\partial c}}^{=1} \underbrace{\frac{\partial c}{\partial b}}_{=1} = \frac{1}{2\sqrt{c}}$$

Solving diff eq with NN.

Example

$$\frac{dg}{dt} = -\gamma g(t)$$

$\gamma > 0 \qquad t \in [0, \infty)$

$g_0 = g(t=0)$

$$g(t) = g_0 \exp(-\gamma t) \Rightarrow$$

$$f(t, g(t), g'(t)) =$$

$$\frac{dg}{dt} - (-\gamma g(t)) = 0$$

Define a new function

$$g_t(t, \theta) = h_1(t) + \underbrace{h_2(t, \theta)}_{t \cdot NN(t, \theta)}$$

$$g_T(t,\theta) = h_\theta(t)$$

$$\quad\quad\quad\;\; \shortparallel$$

$$\quad\quad\quad\;\; g_0 + t\,NN(t,\theta)$$

initial

$$g(t=0) = g_0$$

conditions

$$g_E(t=0) = g_0$$

$$c(t,\theta) = \frac{1}{n}\sum_{i=0}^{n-1} f\left(t, g_T(t,\theta), g_T'(t,\theta)\right)^2$$

Example 2

Two-point boundary

$$-g''(x) = h(x)$$

known

$x \in [0,1]$ $\quad$ $g(x=0) =$

$\quad$ $g(x=1) = 0$

$$g_T(x,\theta) = x(1-x)\,NN(x,\theta)$$

$x=0$ then $g_T(0,\theta) = 0$

$x=1$ — , — —— $= 0$